IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
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N06/5/MATHL/HP3/ENG/TZ0/XX

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MATHEMATICS
HIGHER LEVEL
PAPER 3

Monday 13 November 2006 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions in one section.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Statistics and probability

1. *[Maximum mark: 11]*

Doctor Tosco claims to have found a diet that will reduce a person's weight, on average, by 5 kg in a month. Doctor Crocci claims that the average weight loss is less than this. Ten people use this diet for a month. Their weights before and after are shown below.

Person	A	B	C	D	E	F	G	H	I	J
Weight before (kg)	82.6	78.8	83.1	69.9	74.2	79.5	80.3	76.2	77.8	84.1
Weight after (kg)	75.8	74.1	79.2	65.6	72.2	73.6	76.7	72.9	75.0	79.9

(a) State suitable hypotheses to test the doctors' claims. *[2 marks]*

(b) Use an appropriate test to analyse these data. State your conclusion at

(i) the 1 % significance level;

(ii) the 10 % significance level. *[8 marks]*

(c) What assumption do you have to make about the data? *[1 mark]*

2. *[Maximum mark: 12]*

The random variable X is normally distributed with mean μ and standard deviation 2.5. A random sample of 25 observations of X gave the result

$$\sum x = 315 .$$

(a) Find a 90 % confidence interval for μ . *[6 marks]*

(b) It is believed that $\mathrm{P}(X \leq 14) = 0.55$. Determine whether or not this is consistent with your confidence interval for μ . *[6 marks]*

3. *[Maximum mark: 14]*

A toy manufacturer makes a cubical die with the numbers 1, 2, 3, 4, 5, 6 respectively marked on the six faces. The manufacturer claims that, when it is thrown, the probability distribution of the score X obtained is given by

$$\mathrm{P}(X = x) = \frac{x}{21} \text{ for } x = 1, 2, 3, 4, 5, 6 .$$

To check this claim, Pierre throws the die 420 times with the following results.

x	Frequency
1	25
2	46
3	64
4	82
5	99
6	104

State suitable hypotheses and using an appropriate test determine whether or not the manufacturer's claim can be accepted at the 5 % significance level. *[14 marks]*

4. *[Maximum mark: 23]*

A chocolate manufacturer puts gift vouchers at random into 15 % of all chocolate bars produced. A customer must collect five vouchers to qualify for a gift.

(a) Barry goes into a shop and buys 20 of these bars. Find the probability that he qualifies for a gift. *[3 marks]*

(b) John goes into a shop and buys n of these bars. Find the smallest value of n for which the probability of qualifying for a gift exceeds $\frac{1}{2}$. *[4 marks]*

(c) Martina goes into a shop and buys these bars one at a time: she opens them to see if they contain a voucher. She obtains her 5th voucher on the Xth bar bought.

(i) Write down an expression for $\mathrm{P}(X = x)$, valid for $x \ge 5$.

(ii) Calculate $\mathrm{E}(X)$.

(iii) Show that $\dfrac{\mathrm{P}(X = x)}{\mathrm{P}(X = x-1)} = \dfrac{0.85(x-1)}{x-5}$.

(iv) Show that if $\mathrm{P}(X = x) > \mathrm{P}(X = x-1)$ then $x < \frac{83}{3}$. Deduce the most probable value of X. *[16 marks]*

SECTION B

Sets, relations and groups

1. *[Maximum mark: 9]*

Let A and B be subsets of the set U and let $C = A \cap B$, $D = A' \cup B$ and $E = A \cup B$.

(a) Draw separate Venn diagrams to represent the sets C, D and E. *[3 marks]*

(b) Using de Morgan's laws, show that $A = D' \cup C$. *[3 marks]*

(c) Prove that $B = D \cap E$. *[3 marks]*

2. *[Maximum mark: 11]*

Consider the following groups of order 4:

$G = (\{1, 3, 5, 7\}, \bullet)$ where $\bullet$ is multiplication modulo 8.
$H = (\{3, 6, 9, 12\}, *)$ where $*$ is multiplication modulo 15.

(a) (i) Copy and complete the Cayley table for G.

•	**1**	**3**	**5**	**7**
1				
3			7	
5	5			
7				1

(ii) Draw the Cayley table for H. *[6 marks]*

(b) Determine whether or not they are isomorphic, giving appropriate reasons. *[5 marks]*

3. *[Maximum mark: 17]*

Consider the relations R_1, R_2, R_3 and R_4, represented by the following tables

	a	b	c
a		1	1
b	1	1	1
c	1	1	1

R_1

	A	B	C
A	1		1
B	1	1	
C	1		1

R_2

	d	e	f	g
d	1		1	1
e		1	1	
f	1	1	1	1
g	1		1	1

R_3

	D	E	F	G
D	1		1	
E		1		1
F	1		1	
G		1		1

R_4

(Note that a "1" in the table means that the element in that row is related to the element in that column, *e.g.* in R_2, B is related to A, but A is not related to B.)

(a) For each relation, determine whether or not it is an equivalence relation. In each case, justify your answer. *[15 marks]*

(b) For those which are equivalence relations, describe the corresponding equivalence classes. *[2 marks]*

4. *[Maximum mark: 16]*

Consider the following functions.

$$f : \mathbb{R}^+ \to \mathbb{R}^+ \text{ where } f(x) = x^2 + 3x + 2$$
$$g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R} \text{ where } g(x,\ y) = (3x + 2y,\ 2x + y)$$
$$h : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+\times\mathbb{R}^+ \text{ where } h(x,\ y) = (x + y,\ xy)$$

(a) Explain why f is not surjective. *[2 marks]*

(b) Explain why g has an inverse, and find g^{-1}. *[9 marks]*

(c) Determine, with reasons, whether h is

(i) injective;

(ii) surjective. *[5 marks]*

5. *[Maximum mark: 7]*

The order of each of the elements of the group $(G, *)$ is either 1 or 2. Show that G is an Abelian group. *[7 marks]*

SECTION C

Series and differential equations

1. *[Maximum mark: 9]*

Consider the series $S = \sum_{n=1}^{\infty} \frac{1}{n!}$.

(a) Use the ratio test to prove that this series is convergent. *[4 marks]*

(b) Use a comparison test to show that $S < 2$. *[4 marks]*

(c) Write down the **exact** value of *S*. *[1 mark]*

2. *[Maximum mark: 17]*

(a) Show that the polynomial approximation for $\ln x$ in the interval $[0.5, 1.5]$ obtained by taking the first three non-zero terms of the Taylor series about $x = 1$ is given by

$$\ln x \approx \frac{x^3}{3} - \frac{3x^2}{2} + 3x - \frac{11}{6}.$$ *[7 marks]*

(b) Given $\int \ln x \,\mathrm{d}x = x \ln x - x + C$, show by integrating the above series that another approximation to $\ln x$ is given by

$$\ln x \approx \frac{x^3}{12} - \frac{x^2}{2} + \frac{3x}{2} - \frac{5}{6} - \frac{1}{4x}.$$ *[6 marks]*

(c) Which is the better approximation when $x = 1.5$? *[4 marks]*

3. *[Maximum mark: 19]*

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{1+x}{1-x}\right)\right)=\frac{2}{1-x^2}$, $|x|<1$. *[3 marks]*

(b) Find the solution to the homogeneous differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy-y^2\text{, given that } y=\frac{1}{2} \text{ when } x=1.$$

Give your answer in the form $y=g(x)$. *[16 marks]*

4. *[Maximum mark: 15]*

(a) (i) Find $I_n=\int_{-n}^{\alpha n}\frac{x\,\mathrm{d}x}{1+x^2}$, where α is a positive constant and n is a positive integer.

(ii) Determine $\lim_{n\to\infty} I_n$. *[6 marks]*

(b) Using l'Hôpital's Rule find

$$\lim_{x\to 0}\left(\frac{\tan\beta x-\beta\tan x}{\sin\beta x-\beta\sin x}\right),$$

where β is a non-zero constant different from ± 1. *[9 marks]*

SECTION D

Discrete mathematics

1. *[Maximum mark: 18]*

Consider the following adjacency matrices for the graphs G_1 and G_2:

$$G_1 = \begin{array}{c c} & \begin{array}{ccccc} p & q & r & s & t \end{array} \\ \begin{array}{c} p \\ q \\ r \\ s \\ t \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad G_2 = \begin{array}{c c} & \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw the graphs of G_1 and G_2. *[4 marks]*

(b) For each graph, giving a reason, determine whether or not it

(i) is simple;

(ii) is connected;

(iii) is bipartite;

(iv) is a tree;

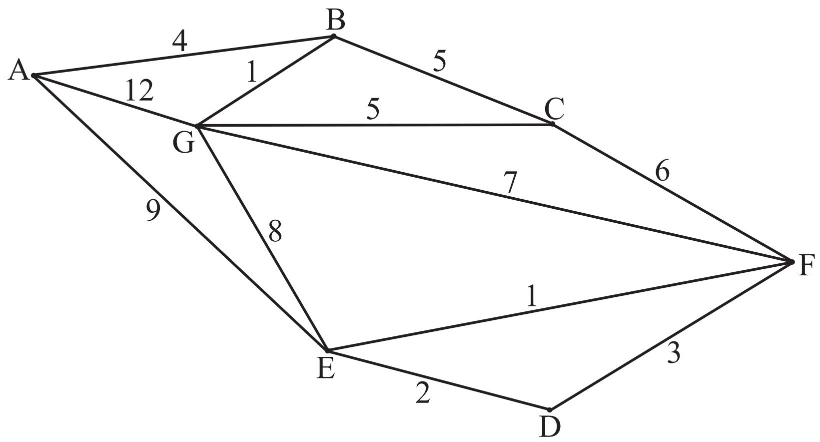
(v) has an Eulerian trail, giving an example of a trail if one exists. *[14 marks]*

2. *[Maximum mark: 10]*

Solve the equation $-38x + 26y = 8$, where $x, y \in \mathbb{Z}$. *[10 marks]*

3. *[Maximum mark: 11]*

The following diagram shows a graph *H*.



(a) Use Kruskal's algorithm to find a minimum spanning tree for *H*. *[9 marks]*

(b) Write down the weight of the minimum spanning tree found. *[2 marks]*

4. *[Maximum mark: 8]*

(a) Prove that a tree is a simple graph. *[3 marks]*

(b) (i) *G* is a complete bipartite graph and graph *W* is the complement of *G*. Prove that *W* is not connected.

(ii) Show by giving an example that the converse is not true. *[5 marks]*

5. *[Maximum mark: 13]*

Fermat's theorem states that a prime number p is a divisor of $x^p - x$ and $y^p - y$, where $x,\ y \in \mathbb{Z}^+$. Show that if $p \mid x^p + y^p$, $p > 2$ then $p^2 \mid x^p + y^p$. *[13 marks]*